Complex Series

• Power series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

• Taylor Series

A power series with nonnegative power terms.

• Laurant Series

A power series with positive and negative power terms.

• Circle of Convergence

Every Complex power series has radius of Convergence R. Amlogous to Concept of an interval of Convergence in real calculus. When $0<R<\infty$, a Complex power series has a Circle of Convergence defined by.

$$|z-z_0| = R$$

• $|z-z_0| < R$ → Power series is Convergent

• $|z-z_0| > R$ → Power series is divergent

Radius of Convergence "R"

$$R = |z_0 - z^*|$$

z^* → the nearest singular pt. to center of series z_0

Scanned by CamScanner

• Taylor Series

let $f(z)$ be analytic within a domain D and let z_0 be a point in D. Then $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Valid for the largest circle "C" with center at z_0 and radius R that lies entirely within D.

• Maclaurin Series

A Taylor series with center $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

• Some special series

1- $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots\cdots \qquad |z|<1$

2- $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots\cdots \qquad |z|<1$

3- $e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \qquad |z|<\infty$

4- $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots \qquad |z|<\infty$

5- $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^{n-1} \frac{z^{2n-2}}{(2n-2)!} + \cdots \qquad |z|<\infty$

6- $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n-1}}{(2n-1)!} + \cdots$

7- $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n-2}}{(2n-2)!} + \cdots$

8- $(1+z)^m = 1 + mz + \frac{m(m-1)}{2!} z^2 + \cdots \qquad |z|<1$

Scanned by CamScanner

1- Find the Maclaurin expansion of $f(z) = \frac{1}{1+z^2}$ and hence Fin it For $\arctan z$.

Sol.

• We know that

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

• So

$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

• Since

$$f(z) = \arctan(z) = \int \frac{1}{1+z^2}\, dz$$

$$\therefore f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots + C$$

• But

$$f(0) = 0 \implies C = 0$$

• Then

$$f(z) = \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

2- Find the Maclaurin expansion of

$$f(z) = \frac{1}{(1-z)^2}$$

Sol.

• we know that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

• by differentiating both sides then

$$f(z) = \frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

Scanned by CamScanner

3- Find Maclaurin expansion of $f(z)$

a) $f(z) = \ln(1+z)$

Sol.

• We know that

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

• Then

$$\ln(1+z) = \int \left(\frac{1}{1+z}\right) dz$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots + C$$

$f(0) = 0 \Rightarrow C = 0$

$$\therefore \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

b) $f(z) = \ln(1-z)$

Sol.

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

c) $f(z) = \ln\left(\frac{1+z}{1-z}\right)$

Sol.

$$f(z) = \ln(1+z) - \ln(1-z)$$

$$\therefore f(z) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

Scanned by CamScanner

4- Without actually expanding determine the radius of convergence of the taylor series of the given function centered at the indicated point.

a) $f(z) = \frac{2i}{4+iz}$ $z_0 = -3i$

Sol.

• The singular point is $z^* = 4i \Rightarrow R = |z_0 - z^*| = |-3i - 4i|$

$$R = |-7i| \Rightarrow R = 7$$

b) $f(z) = \frac{4+5z}{1+z^2}$ $z_0 = 2+5i$

Sol.

$$f(z) = \frac{4+5z}{(z+i)(z-i)}$$

• two singular points

$$z_1 = i \quad \text{and} \quad z_2 = -i$$

• For $z = i$

$$R = |2+5i-i| = |2+4i| = \sqrt{20}$$

• For $z = -i$

$$R = |2+5i+i| = |2+6i| = \sqrt{40}$$

Thus, the radius of convergence $R = \sqrt{20}$

c) $f(z) = \cot z$ $z_0 = \pi i$

Sol.

• Singular pts. $\sin z = 0 \Rightarrow z = n\pi,\ n = 0, \pm 1, \pm 2, \ldots$

$$\therefore R = |z_0 - z^*| = |\pi i - 0| = |\pi i| = \pi$$

Scanned by CamScanner

Laurant Series

A series representation of a function $f(z)$ that has the form

$$\boxed{f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{1}{2\pi i}\oint \frac{f(s)}{(s-z_0)^{n+1}}\,ds}$$

- $$f(z) = \cdots + \overbrace{\frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}}^{\text{Principle part}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2}_{\text{analytic part}} + \cdots$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\left|\frac{1}{z-z_0}\right| < r_1 \qquad\qquad |z-z_0| < r_2$$

$$\Downarrow$$

$$\boxed{r_1 < |z-z_0| < r_2}$$

5- Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurant series valid for

1) $0 < |z| < 1$

Sol.

$$f(z) = \frac{-1}{z}\left(\frac{1}{1-z}\right)$$

$$= \frac{-1}{z}\left(1 + z + z^2 + \cdots\right)$$

$$\therefore \boxed{f(z) = \frac{-1}{z} - 1 - z - z^2 - \cdots}$$

Scanned by CamScanner

2- $|z|>1$

Sol.

$$|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$$

$$\therefore f(z) = \frac{1}{z^2}\left(1/(1-\tfrac{1}{z})\right)$$

$$= \frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$\therefore f(z) = \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots$$

3- $0<|z-1|<1$

Sol.

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{(z-1)}\cdot\frac{1}{(1+(z-1))}$$

$$= \frac{1}{(z-1)}\cdot\left(1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right)$$

$$f(z) = \frac{1}{z-1}-1+(z-1)-(z-1)^2+(z-1)^3-\cdots$$

4- $|z-1|>1$

Sol.

$$|z-1|>1 \Rightarrow \left|\frac{1}{z-1}\right|<1$$

$$\therefore f(z) = \frac{1}{(z-1)^2}\cdot\left(\frac{1}{(1+\frac{1}{z-1})}\right)$$

$$= \frac{1}{(z-1)^2}\cdot\left(1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right)$$

$$f(z) = \frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\frac{1}{(z-1)^5}+\cdots$$

Scanned by CamScanner

6- Expand $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ in a laurant series valid for

1) $0<|z-1|<2$

Sol.

$$f(z) = \frac{1}{(z-1)^2} \cdot \frac{1}{((z-1)-2)} = \frac{-1}{2(z-1)^2} \cdot \left(\frac{1}{1-\left(\frac{z-1}{2}\right)}\right)$$

★ $\left|\frac{z-1}{2}\right| < 1 \Rightarrow |z-1|<2$

$$\therefore f(z) = \frac{-1}{2(z-1)^2} \cdot \left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{4} + \frac{(z-1)^3}{8} + \cdots\right]$$

2) $0<|z-3|<2$

Sol.

$$f(z) = \frac{1}{2^2(z-3)} \cdot \frac{1}{\left(1+\frac{z-3}{2}\right)^2} = \frac{1}{4(z-3)} \cdot \left(1+\frac{z-3}{2}\right)^{-2}$$

★Binomial theorem

$$\left(1+\frac{z-3}{2}\right)^{-2} = 1 + (-2)\left(\frac{z-3}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2 + \cdots$$

$$\therefore f(z) = \frac{1}{4(z-3)} \cdot \left(1 - (z-3) + 3\left(\frac{z-3}{2}\right)^2 + \cdots\right)$$

7- Expand $f(z) = \dfrac{1}{z(z-1)}$ in a laurant series valid for

$1<|z-2|<2$

Sol.

Using partial fractions

$$\therefore f(z) = \frac{-1}{z} + \frac{1}{z-1}$$

$$= f_1(z) + f_2(z)$$

Scanned by CamScanner

$\bullet\ F_1(z) = \frac{-1}{z}$

$$= \frac{-1}{2}\left(\frac{1}{1+\frac{z-2}{2}}\right) = \frac{-1}{2}\left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \cdots\right)$$

$\bullet\ F_2(z) = \frac{1}{z-1}$

$$= \frac{1}{1+(z-2)} = \frac{1}{(z-2)} \cdot \left(\frac{1}{1+\frac{1}{z-2}}\right)$$

$$= \frac{1}{z-2} \cdot \left(1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-3)^3} + \cdots\right)$$

$$\therefore F(z) = F_1(z) + F_2(z)$$

8- Expand $F(z) = \frac{z^2+2z+2}{z-2}$ in laurant series valid for

$\bullet\ |z-2| > 0$

Sol.

$$\therefore F(z) = z + 4 + \frac{10}{z-2}$$

$$\begin{array}{r|l} & z+4 \\ \hline z-2 & z^2+2z+2 \\ & -(z^2-2z) \\ \hline & 4z+2 \\ & -(4z-8) \\ \hline & 10 \end{array}$$

$$\therefore F(z) = (z-2) + 6 + \frac{10}{z-2}$$

$\bullet\ |z-1| > 1$

Sol.

$$= (z-1) + 5 + \frac{10}{z-2}$$

$$= (z-1) + 5 + \frac{1}{(z-1)} \cdot \left(\frac{10}{1-\frac{1}{z-1}}\right) = (z-1) + 5 + \frac{10}{(z-1)}\left(\frac{1}{1-\frac{1}{z-1}}\right)$$

$$F(z) = (z-1) + 5 + \frac{10}{z-1}\left(1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} \cdots\right)$$

Scanned by CamScanner

9- Find laurant series representation for

$$f(z) = \frac{\cos z - 1}{z^4}$$ that involves powers of z

Sol.

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\therefore \cos z - 1 = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\therefore f(z) = \frac{-1}{2z^2} + \frac{1}{24} - \frac{z^2}{6!} + \cdots$$

10- Expand

$$f(z) = e^{3/z}$$ in alaurant series valid for $|z| > 0$

Sol.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!\,z^2} + \frac{3^3}{3!\,z^3} + \cdots$$

This series is valid for $|z| > 0$

Scanned by CamScanner